One-Sided Terrain Guarding and Chordal Graphs

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Abstract. The TERRAIN GUARDING problem, a variant of the famous ART GALLERY problem, has garnered significant attention over the last two decades in Computational Geometry from the viewpoint of complexity and approximability. Both the continuous and discrete versions of the problem were shown to be NP-Hard in [23] and to admit a PTAS [24,14]. The biggest unsolved question regarding this problem is if it is fixed-parameter tractable with respect to the size of the guard set. In this paper, we present two theorems that establish a relationship between a restricted case of the ONE-SIDED TERRAIN GUARDING problem and the CLIQUE COVER problem in chordal graphs. Similar results were obtained in [20] for a special class of terrains called orthogonal terrains and were used to present a FPT algorithm with respect to the parameter that we require for DISCRETE ORTHOGONAL TERRAIN GUARDING in [3]. We hope that the results obtained in this paper can, in future work, be used to produce such an algorithm for DISCRETE TERRAIN GUARDING.

Keywords: Terrain Guarding · Chordal Graphs · Visibility Graphs.

1 Introduction

Let $V = \{v_1, \ldots, v_n\}$ be a finite sequence of three or more points in $\mathbb{R}^2$. The polygonal chain defined by V is the curve specified by the line segments connecting v_i and v_{i+1} for all $1 \leq i < n$. In this paper, we additionally assume that polygonal chains are simple curves. For a point v in $\mathbb{R}^2$, we use $x(v)$ and $y(v)$ to denote the x and y coordinates of v. A *1.5-dimensional terrain* (which we will refer to as a *terrain*) is a polygonal chain defined by V where $x(v_i) \leq x(v_j)$ for all i and j such that $1 \leq i < j \leq n$. We say that two points t_1 and t_2 on a terrain T *see* or *guard* each other if no point on $\overline{t_1 t_2}$, the line segment joining t_1 and t_2, lies strictly below the terrain. Examples of terrains are shown in Figures 1a and 1b. Let U be a set of points on the terrain. The *visibility region* of U is defined to be the collection of all points on the terrain which is seen by at least one point of U. We let VIS U denote this set. The encircled vertices in Figure 1a are precisely the ones that are present in VIS U when $U = \{v_2, v_9\}$. When U contains a single element, say u, we abuse this notation and write VIS u instead of VIS U. It is sometimes useful to view a terrain T as an undirected graph with vertices V and edges $E = \{(v_i, v_{i+1}) \mid 1 \leq i < n\}$. We switch between viewing a terrain as a polygonal chain and a graph frequently in this paper.

These definitions naturally lead us to the three major versions of the terrain guarding problem. They revolve around finding k-many points (called *guards*) on the terrain to guard a chosen set of points of the terrain. In the CONTINUOUS TERRAIN GUARDING problem, we are required to guard the entire terrain by placing guards anywhere on the terrain. In the DISCRETE TERRAIN GUARDING version, we are only required to guard the vertex set but can only place guards on the vertices themselves. ANNOTATED TERRAIN GUARDING generalizes the discrete version by restricting the vertices where the guards can be placed to a subset of V while requiring us to guard a given subset of vertices. We define this problem formally below (referenced from [3]). Hereafter, we assume that the number of vertices of a terrain is n.

Problem. ANNOTATED TERRAIN GUARDING: Let $T(V, E)$ be a terrain, $k \in \mathbb{N}$ and $\mathcal{G}, \mathcal{C} \subseteq V$. Decide if there exists a $S \subseteq \mathcal{G}$ with $|S| \leq k$ such that VIS $S \supseteq \mathcal{C}$.

Note that if $\mathcal{G} = \mathcal{C} = V$ in the annotated version of the problem, then it is exactly the DISCRETE TERRAIN GUARDING problem. We use $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ to denote an instance of the ANNOTATED TERRAIN GUARDING problem. The *visibility graph* of such an instance, G_T, is defined to be the undirected graph $G_T = (\mathcal{C}, E')$, where $E' = \{(u, v) \in \mathcal{C}^2 \mid \text{there is a } g \in \mathcal{G} \text{ that sees } u \text{ and } v\}$. In some variants of the ART GALLERY problem, vertices in the visibility graph are connected by an edge if those two vertices see each other [26,17]. Terrain visibility graphs, when defined as above, have been studied previously - for example, in [12,2]. Here, however, there exists an edge between two vertices of G_T if there exists an element in the guard set which can see both these vertices.

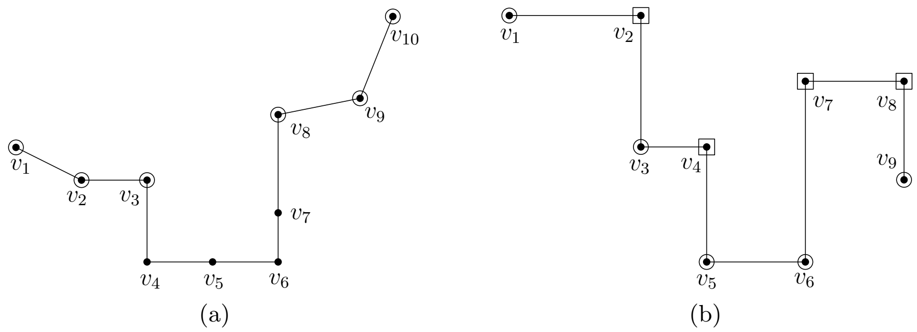

Fig. 1. Examples of terrains where the vertices and edges are marked by small discs and straight lines respectively. In (a), the vertices that are seen by $U = \{v_2, v_9\}$ are encircled. The second figure is an example of an orthogonal terrain.

A subclass of terrains which are of particular interest are orthogonal terrains. In an *orthogonal* (or *rectilinear*) terrain, each edge is either parallel to the x-axis or parallel to the y-axis. Furthermore, each vertex is incident to at most one edge of each type. An example of an orthogonal terrain is given in Figure 1b.

A graph $G(V,E)$ is *chordal* if for any $V' \subseteq V$, where $|V'| \geq 4$, the subgraph induced by V' is not a cycle. Equivalently, G is chordal if the graph induced by any cycle of length at least 4 is not a cycle. Chordal graphs have been well studied in literature since a lot of the typical NP-Hard graph problems can be solved quickly for this graph class [18]. In particular, there exists a simple polynomial time algorithm which solves the Clique Cover problem in chordal graphs [15]. The Clique Cover problem is defined as follows: given a graph $G(V,E)$ and a $k \in \mathbb{N}$, decide if there exists a collection of k-many cliques of G that covers V. An instance of this problem is denoted by $(G(V,E), n, k)$ where $n = |V|$.

In the field of Parameterized Complexity, each instance I of a problem $\mathcal{P}$ is associated with a parameter k. $\mathcal{P}$ is said to be *fixed-parameter tractable* (FPT) if there exists an algorithm $\mathcal{A}$, a computable function f defined on $\mathbb{N}$, and a constant c such that given an instance (I,k) of $\mathcal{P}$, $\mathcal{A}$ decides correctly in $\mathcal{O}(f(k)\cdot |I|^c)$ time if (I,k) is a Yes instance of $\mathcal{P}$. Here, $|I|$ denotes the size of the input instance I. Such an algorithm $\mathcal{A}$ is called a *fixed-parameter algorithm* or a *FPT algorithm.* For a comprehensive introduction into Parameterized Complexity and FPT algorithms, we refer the reader to the following books [10,9].

1.1 Motivation

Optimal guarding of terrains arises in the placement of antennas for communication networks. We study this problem in two dimensions to understand better the considerably more difficult problem of guarding terrains in three dimensions. Moreover, 1.5-dimensional terrains arise directly in applications of coverage along a highway as well as in security lamp and camera placement along walls and streets [5,20,23].

1.2 Related Work

Art Gallery is a classical problem in Computational Geometry and has been studied extensively over the last five decades. In this problem, we are given a polygon and are asked to find the minimum set of guards required to guard a specified set of points of the polygon. We say a point *guards* another if the line segment joining them lies within the polygon. Since it is impossible to survey here the vast literature that discusses various versions and results regarding this problem, we refer the reader to books, surveys and chapters dedicated to it [26,27,16,6].

The Terrain Guarding problem was stated in 1995 by Chen *et al.* in [7]. In the same paper, the authors hypothesized that both the continuous and discrete versions of the problem are NP-Hard, but did not provide a concrete proof in support of their claim. It was only in 2010 that King and Krohn finally showed that both the Continuous Terrain Guarding and Discrete Terrain Guarding problems are NP-Hard [23]. Meanwhile, the problem continued to be studied from the viewpoint of approximation algorithms and Ben-Moshe *et al.* [5] proposed the first constant-factor approximation for the discrete version of the problem. The factor of approximation was improved over the course of several papers [22,8,11] and finally a PTAS for the discrete version of the problem was given by Krohn *et al.* in 2014 [24]. A PTAS for Continuous Terrain Guarding was obtained a couple of years later by Friedrichs *et al.* [14]. This work also proved that the continuous version of the problem is NP-Complete.

Thus, we have a satisfactory understanding of the approximability of the terrain guarding problem. In the paper that they proved the NP-Hardness of the terrain guarding problems, King and Krohn stated that the biggest remaining question regarding this problem was its fixed-parameter tractability. Terrain guarding has been shown to have a FPT algorithm with respect to few parameters [21,1] but it is still not known if the problem is fixed-parameter tractable with respect to the number of guards that are required to guard the terrain. In 2018, Ashok *et al.* showed that this is indeed true for the Discrete Orthogonal Terrain Guarding problem in [3]. Their algorithm exploited a connection between guarding orthogonal terrains and covering chordal graphs with cliques that was established by Katz and Roisman in Lemmas 3.6 and 3.7 of their paper [20]. In these lemmas, they considered the visibility graph of the Annotated Orthogonal Terrain Guarding instance $(T(V,E), n, k, \mathscr{R}, \mathscr{C}_l)$ and proved that it is chordal (the sets $\mathscr{R}$ and $\mathscr{C}_l$ are defined in Section 2). They then showed that any clique of the visibility graph can been seen by a single guard. In this paper, we will show that these lemmas can be stated and proved for a special case of the annotated version of the terrain guarding problem called the Left-Sided Terrain Guarding problem. Informally, in this version of the problem, the guards can only see to their left.

A preliminary version of this paper appeared at CALDAM-2021 [25]. In this final version, the proof of Theorem 3.1 has been substantially shortened. Moreover, Section 4, which explores two applications of our main result, has been added. The statements of some of the results presented in Section 3 have been modified to be more concise.

1.3 Results

This paper presents two theorems which prove the equivalence between a restricted case of the Left-Sided Terrain Guarding problem and the Clique Cover problem in chordal graphs. Theorem 3.1 proves that the visibility graph corresponding to an instance of this problem is chordal. Theorem 3.2 builds on top of this and proves that there exists a clique in the visibility graph, if, and only if, there exists a guard that sees all the vertices of that clique. Collating these two theorems gives us the main result of this paper. Lemmas 3.4 and 3.5 show that this paper indeed generalizes results that are known for orthogonal terrains.

Main Result Let $(T(V,E), n, k, \mathcal{G}, \mathcal{C})$ be a Left-Sided Terrain Guarding instance where LVis $\mathcal{G} \supseteq \mathcal{C}$. Then, this is a true instance of the problem if, and only if, $(G_T(\mathcal{C}, E'), |\mathcal{C}|, k)$ is a true instance of the Clique Cover problem where G_T, the left visibility graph of T, is a chordal graph.

We generalize this result in two directions in Section 4. In Section 4.1, we describe a discretization technique to solve the LEFT-SIDED CONTINUOUS TERRAIN GUARDING problem in polynomial time using the main result of this paper. Informally, in this version of the TERRAIN GUARDING problem, we are required to guard the entire terrain using guards that can only see to their left (refer to Section 4.1 for the precise definition). In the final subsection of this paper, we define *terrain-like* graphs and the LEFT-SIDED DOMINATING SET problem. We modify the proofs of the results proved in Section 3 to solve the LEFT-SIDED DOMINATING SET problem in terrain-like graphs (Theorem 4.5, Lemma 4.6).

2 Preliminaries

For points t_1 and t_2 on T, we say t_1 *precedes* t_2, denoted by $t_1 \prec t_2$, if t_1 appears on the terrain before t_2 does (the terrain is scanned from left to right). We say that $t_1 \preceq t_2$ if $t_1 \prec t_2$ or $t_1 = t_2$. The Order Claim, which was originally stated in [5] and later slightly generalized in [1], lays the foundation for the theorems that follow in the next section.

Lemma 2.1 (Order Claim). *Let a, b, c and d be four points on a terrain $T(V, E)$ such that $a \prec b \prec c \prec d$. If a sees c and b sees d, then a sees d.*

In an orthogonal terrain $T(V, E)$, a vertex v_i, where $1 < i < n$, is *convex* if $x(v_i) = x(v_{i+1})$ and $y(v_i) < y(v_{i+1})$ or $x(v_i) = x(v_{i-1})$ and $y(v_i) < y(v_{i-1})$ and is *reflex* otherwise. Equivalently, v_i is a convex vertex if the angle formed by the vertices v_{i-1}, v_i and v_{i+1} (measured above the terrain) is convex and is a reflex vertex otherwise. It is a *left* vertex if $x(v_{i-1}) = x(v_i)$ and a *right* vertex if $x(v_i) = x(v_{i+1})$. The set of convex vertices is denoted by $\mathscr{C}$ and the set of reflex vertices is denoted by $\mathscr{R}$. In Figure 1b, the convex vertices are encircled and the reflex vertices are marked using squares. The set of vertices which are both convex and left are called *left convex* vertices and is denoted by $\mathscr{C}_l$. *Right convex*, *left reflex* and *right reflex* vertices are defined similarly and are denoted by $\mathscr{C}_r$, $\mathscr{R}_l$, and $\mathscr{R}_r$ respectively. Vertices $a \in \mathscr{C}_l$ and $b \in \mathscr{R}_r$ are said to be of the *opposite type* as are vertices $c \in \mathscr{C}_r$ and $d \in \mathscr{R}_l$. v_1 is defined to be of the opposite type as v_2 and v_n is defined to be that of v_{n-1}. In Figure 1b, $\mathscr{R}_l = \{v_7\}$, $\mathscr{R}_r = \{v_2, v_4, v_8\}$, $\mathscr{C}_l = \{v_1, v_3, v_5, v_9\}$ and $\mathscr{C}_r = \{v_6\}$. We now describe the ONE-SIDED TERRAIN GUARDING problem.

Problem. ONE-SIDED TERRAIN GUARDING: Let $T(V, E)$ be a terrain, $k_l, k_r \in \mathbb{N}$ and $\mathcal{G}_l, \mathcal{G}_r, \mathcal{C} \subseteq V$. Decide if there exists a $S_l \subseteq \mathcal{G}_l$ and $S_r \subseteq \mathcal{G}_r$ with $|S_l| \leq k_l$ and $|S_r| \leq k_r$ such that for all $v \in \mathcal{C}$, there is a $g \in S_l$ such that $v \prec g$ and g sees v or a $g \in S_r$ such that $g \prec v$ and g sees v.

We focus on two natural restrictions of the ONE-SIDED TERRAIN GUARDING problem in this paper. In LEFT-SIDED TERRAIN GUARDING, we have $\mathcal{G}_r = \emptyset$ and in RIGHT-SIDED TERRAIN GUARDING, we have $\mathcal{G}_l = \emptyset$. That is, in the left-sided (right-sided) version of the ONE-SIDED TERRAIN GUARDING problem, we must guard $\mathcal{C}$ using guards which can only see to their left (right). We explicitly define the LEFT-SIDED TERRAIN GUARDING problem below.

Problem. LEFT-SIDED TERRAIN GUARDING: Given a terrain $T(V, E)$, $k \in \mathbb{N}$ and $\mathcal{G}, \mathcal{C} \subseteq V$ decide if there exists a $S \subseteq \mathcal{G}$ with $|S| \leq k$ such that for all $v \in \mathcal{C}$, there is a $g \in S$ such that $v \prec g$ and g sees v.

In this case, we say that $\mathcal{G}$ is a set of *left guards*. Moreover, for any $g \in \mathcal{G}$, we let LVIS g - which we call the *left visibility region* of g, to be set of all points on the terrain which lie to left of g and can be seen by g. Similarly, we define the left visibility region of a $U \subseteq \mathcal{G}$ as the union of the left visibility regions of the elements of U. For the RIGHT-SIDED TERRAIN GUARDING problem, *right guards* and *right visibility regions* are defined symmetrically. We use $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ to denote an instance for both these problems. It will be clear from context if the instance corresponds to the annotated version or the left or right-sided versions of the problem. Finally, we define the *left visibility graph* of a LEFT-SIDED TERRAIN GUARDING instance $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ as the undirected graph $G_T(\mathcal{C}, E')$ where $E' = \{(u, v) \in \mathcal{C}^2 \mid \text{there is a } g \in \mathcal{G} \text{ such that } u, v \in \text{LVIS } g\}$. The *right visibility graph* is defined similarly for the RIGHT-SIDED TERRAIN GUARDING problem.

In the paper that they introduced the terrain guarding problem [7], Chen *et al.* also described the left and right-sided versions of the problem. They produced an algorithm, which they called *Army-Withdraw*, which ran in linear time to solve these versions. Elbassioni *et al.* [11] constructed a bipartite graph G from a LEFT-SIDED TERRAIN GUARDING instance $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ where $\mathcal{G} \cap \mathcal{C} = \emptyset$ with the bipartition $(\mathcal{G}, \mathcal{C})$. An element $(g, c) \in \mathcal{G} \times \mathcal{C}$ was an edge of this graph if $c \in \text{LVIS } g$. They then proved that the vertex-vertex incidence matrix corresponding to this graph is *totally balanced* and used the properties of such matrices to produce a 4-approximation algorithm for the ANNOTATED TERRAIN GUARDING problem where $\mathcal{G} \cap \mathcal{C} = \emptyset$. The author refers the reader to [13] and [18] for a detailed discussion on totally balanced matrices.

3 Terrains and Chordal Graphs

In this section, we will prove two theorems which will lead us to the main result of this paper. Even though this section deals exclusively with the LEFT-SIDED TERRAIN GUARDING problem, the claims and their proofs apply, by symmetry, to the RIGHT-SIDED TERRAIN GUARDING problem. The first theorem proves that the left visibility graph of a LEFT-SIDED TERRAIN GUARDING instance $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ is chordal. The proof of this theorem considers a cycle C of length p, where $p \geq 4$, in G_T (the left visibility graph of this instance) and proves that the subgraph induced by C, denoted by $G_T[C]$, is not a cycle. This is done by using Lemma 2.1 on the various cases that arise depending on the positions of the vertices of C and the guards that see them on the terrain.

The second theorem considers a LEFT-SIDED TERRAIN GUARDING instance $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ where LVIS $\mathcal{G} \supseteq \mathcal{C}$. It proves that the vertices of any clique of G_T can be seen by a single guard. This proves that k-many guards can see all of $\mathcal{C}$ if, and only if, there exists k-many cliques that cover G_T. This, along with the previous theorem, directly proves our main result. We prove this theorem using induction over the number of vertices in the clique. In practice, the additional assumption that $\mathcal{G}$ sees all of $\mathcal{C}$ is very weak - if $\mathcal{G}$ does not see $\mathcal{C}$, then, for any natural number k, $(T(V,E), n, k, \mathcal{G}, \mathcal{C})$ is a false instance of the LEFT-SIDED TERRAIN GUARDING problem anyway.

We then prove an important corollary of our main result. It states that if a LEFT-SIDED TERRAIN GUARDING instance is false, then there exists a small subset of $\mathcal{C}$ (with $k+1$ vertices) that cannot be seen by k-many guards. We prove this by producing an independent set U of size $k+1$ in G_T and observing that if k guards did see all the vertices of U, then U would fail to be an independent set. Finally, we draw parallels between our results and the following lemma due to Katz and Roisman [20]: for an orthogonal terrain T, $(T(V,E), n, k, \mathscr{R}, \mathscr{C}_l)$ is a true instance if, and only if, $(G_l(\mathscr{C}_l, E'), |\mathscr{C}_l|, k)$ is a true instance of the CLIQUE COVER problem. This is done by observing that left convex vertices can only see to one side.

Theorem 3.1. *Let $(T(V,E), n, k, \mathcal{G}, \mathcal{C})$ be a* LEFT-SIDED TERRAIN GUARDING *instance. Then, the left visibility graph of this instance, say G_T, is chordal.*

Proof. Let $C \subseteq \mathcal{C}$ where $|C| = p \geq 4$ be a cycle in G_T. We prove that $G_T[C]$ is not a cycle. Let $C = \{c_1, c_2 \ldots c_p\}$ be the order of the vertices as they appear on the cycle. Also, we assume, without loss in generality, that $c_i \preceq c_1$ for all $c_i \in C$ and that $c_p \prec c_2$. As c_1 and c_p are neighbours in G_T, there is a left guard $g_{1,p}$ which sees both these vertices. Similarly, we have $g_{1,2}$, a left guard, which sees both c_1 and c_2. Note that $c_1 \prec g_{1,p}$ and $c_1 \prec g_{1,2}$. If $g_{1,2} = g_{1,p} = g$, then g sees both c_2 and c_p. This implies that c_2 and c_p share an edge in $G_T[C]$. Since $p \geq 4$, (c_2, c_p) is a chord of the cycle. Thus, $G_T[C]$ is not a cycle. We are now left with two cases:

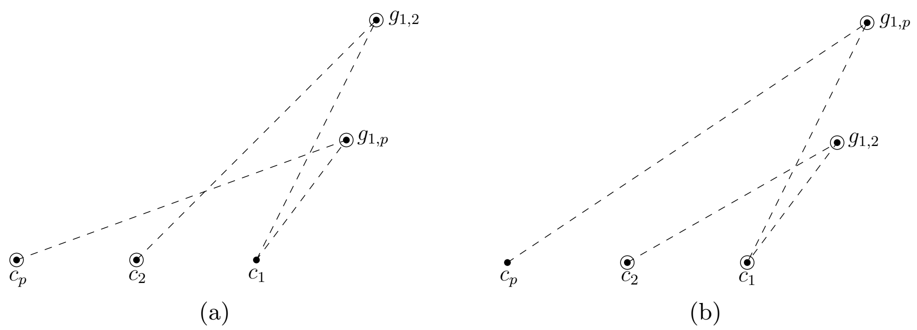

Fig. 2. Part (a) illustrates Case 3.1.1 of Theorem 3.1. Here, $g_{1,p}$ precedes $g_{1,2}$. Two vertices that see each other are connected by a dashed line. If we substitute a, b, c and d with c_p, c_2, $g_{1,p}$ and $g_{1,2}$ respectively in Lemma 2.1, we get that c_p sees $g_{1,2}$ in this case. Part (b) depicts Case 3.1.2 where $g_{1,2} \prec g_{1,p}$. We can apply Lemma 2.1 on the encircled vertices. On doing so, we get that c_2 sees $g_{1,p}$.

Case 3.1.1 ($g_{1,p} \prec g_{1,2}$). This is illustrated in Figure 2a. In this case, we have $c_p \prec c_2 \prec g_{1,p} \prec g_{1,2}$ and c_p sees $g_{1,p}$ while c_2 sees $g_{1,2}$. Thus, by Lemma 2.1, $g_{1,2}$ sees c_p. Since $g_{1,2}$ sees c_2 by construction, there is an edge between c_2 and c_p in G_T. As observed previously, this implies that $G_T[C]$ is not a cycle.

Case 3.1.2 ($g_{1,2} \prec g_{1,p}$). Figure 2b illustrates this case. By construction, we have $c_2 \prec c_1 \prec g_{1,2} \prec g_{1,p}$ and c_2 sees $g_{1,2}$ while c_1 sees $g_{1,p}$. We infer that $g_{1,p}$ sees c_2 by applying Lemma 2.1 on these vertices. Thus, there is an edge between c_2 and c_p in $G_T[C]$ proving that $G_T[C]$ is not a cycle.

This completes the proof of this theorem since C was an arbitrary cycle of length at least 4. □

Theorem 3.2. *Let $(T(V,E), n, k, \mathcal{G}, \mathcal{C})$ be a* LEFT-SIDED TERRAIN GUARDING *instance where* LVIS $\mathcal{G} \supseteq \mathcal{C}$. *Then, if G_T denotes the left visibility graph of this instance, for $K \subseteq \mathcal{C}$, $G_T[K]$ is a clique if, and only if, there is a $g \in \mathcal{G}$ such that* LVIS $g \supseteq K$.

Proof. Let K be a set such that there is a $g \in \mathcal{G}$ such that LVIS $g \supseteq K$. Then, for any pair of vertices in K there is an edge between them in $G_T[K]$ since there is a guard (g itself) seeing them both. Thus, $G_T[K]$ is a clique. Now, we prove the forward direction of the claim. Assume that $K \subseteq \mathcal{C}$ such that $G_T[K]$ is a clique. We prove that there exists a guard seeing all of K by induction on the number of vertices in K. If $|K| = 1$ or $|K| = 2$, then, since LVIS $\mathcal{G} \supseteq \mathcal{C}$, our claim follows trivially. Assume that our supposition holds for all cliques of size at most p, where $p \geq 2$.

Let $K = \{k_1, k_2, \ldots k_p, k_{p+1}\}$ be a subset of $\mathcal{C}$ such that $G_T[K]$ is a clique. The vertices of K are ordered according to how they appear on the terrain. Let $K' = \{k_2, k_3 \ldots k_p, k_{p+1}\}$. Since $G_T[K']$ is a clique of size p, by the induction

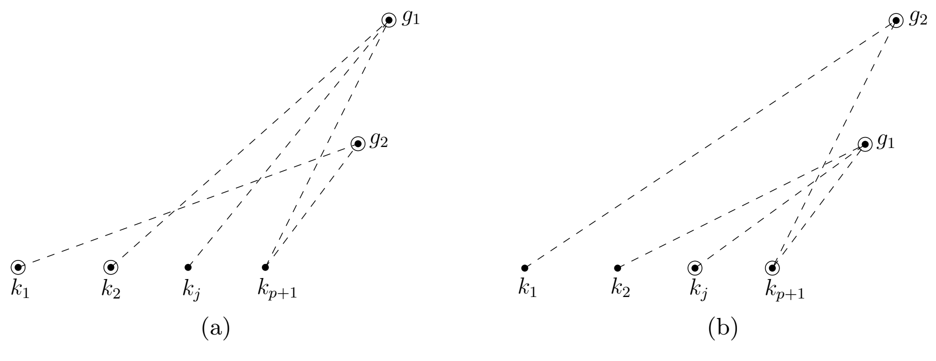

Fig. 3. This figure corresponds to the cases that arise in Theorem 3.2. In (a) g_2 precedes g_1 while in (b) g_1 precedes g_2. By Lemma 2.1, g_1 also sees k_1 in (a) and g_2 guards all the vertices from k_2 through to k_{p+1} in (b).

hypothesis, there is a left guard g_1 such that LVIS $g_1 \supseteq K'$. Since there is a (k_1, k_{p+1}) edge in $G_T[K]$, there is a left guard, say g_2, which sees k_1 and k_{p+1}. If $g_1 = g_2 = g$, then we have LVIS $g \supseteq K$ proving the supposition. We are now left with two cases:

Case 3.2.1 ($g_2 \prec g_1$). This case is shown in Figure 3a. Note that $k_1 \prec k_2 \prec g_2 \prec g_1$ and k_1 sees g_2 while k_2 sees g_1. Hence, by Lemma 2.1, g_1 guards k_1 as well. Thus, LVIS $g_1 \supseteq K$.

Case 3.2.2 ($g_1 \prec g_2$). This is illustrated in Figure 3b. Here, for any $k_j \in K'$ where $j < p + 1$, $k_j \prec k_{p+1} \prec g_1 \prec g_2$. By Lemma 2.1, we have that g_2 sees k_j for all such j. Thus, LVIS $g_2 \supseteq K$.

This proves our supposition and completes the proof by induction. Note that the situations illustrated in Figures 3a and 3b are similar to the ones in Figures 2a and 2b. They are presented in this proof again for clarity. □

Theorems 3.1 and 3.2 are, to the best of the author's knowledge, an addition to existing literature. Combining these two theorems gives us the main result of this paper. As stated at the beginning of this section, a similar result also holds for the RIGHT-SIDED TERRAIN GUARDING problem.

Theorem 3.3. *Let $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ be a* LEFT-SIDED TERRAIN GUARDING *instance where* LVIS $\mathcal{G} \supseteq \mathcal{C}$*. Then, this is a true instance of the problem if, and only if, $(G_T(\mathcal{C}, E'), |\mathcal{C}|, k)$ is a true instance of the* CLIQUE COVER *problem where G_T, the left visibility graph of T, is a chordal graph.*

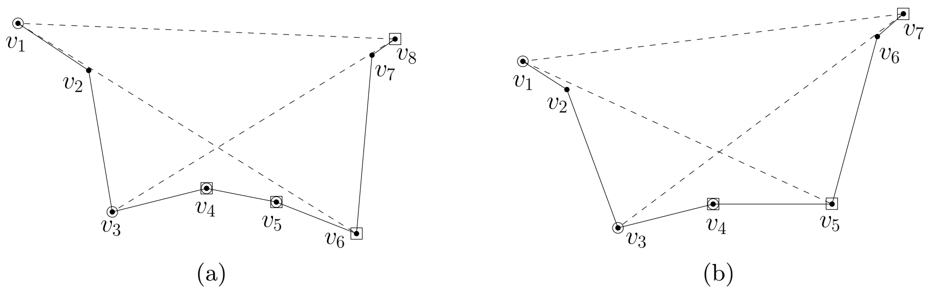

Fig. 4. This figure provides examples of terrains in which Theorem 3.3 fails to be true if left guards are allowed to see themselves. In these two terrains, the vertices of $\mathcal{C}$ and $\mathcal{G}$ have been encircled and marked by squares respectively. In (a), we illustrate a terrain for which Theorem 3.1 is no longer true. Part (b) presents an example where a clique in G_T is not seen by a single left guard.

We take a brief detour to discuss the correctness of Theorem 3.3 if left guards are allowed to see themselves (note that by our definition of LEFT-SIDED TERRAIN GUARDING in Section 1, a left guard cannot see itself). Unfortunately, both Theorems 3.1 and 3.2 are false if this is the case. Examples of terrains that do not satisfy these theorems are presented in Figures 4a and 4b. Consider the terrain in Figure 4a with $\mathcal{G} = \{v_4, v_5, v_6, v_8\}$ as the set of left guards and $\mathcal{C} = \{v_1, v_3, v_4, v_5\}$. Clearly, the visibility graph of this instance is the four cycle. Hence, it is not chordal. In the terrain illustrated by Figure 4b, we let $\mathcal{C} = \{v_1, v_3, v_4\}$ and $\mathcal{G} = \{v_4, v_5, v_7\}$ be a set of left guards. v_1 shares an edge with both v_3 and v_4 in G_T since v_7 sees both v_1 and v_3 while v_5 sees both v_1 and v_4. Furthermore, since v_4 sees itself as well as v_3, there is an edge between v_3 and v_4 in G_T. Thus, $G_T[\mathcal{C}]$ is a clique. However, none of the three guards in $\mathcal{G}$ guard all the vertices of $\mathcal{C}$: v_4 does not see v_1, v_5 does not see v_3, and v_7 does not see v_4. It is also clear that Theorem 3.2 fails to hold if $\mathcal{G}$ does not see all of $\mathcal{C}$. For example, if $\mathcal{C} = \{v_3\}$ and $\mathcal{G} = \{v_5\}$ in the terrain illustrated in Figure 4b, then the isolated vertex v_3 is a clique but no guard in $\mathcal{G}$ sees it.

It is well known that in a chordal graph $G(V, E)$, the minimum number of cliques required to cover V, denoted by $\overline{\chi}(G)$, is equal to the size of a maximum sized independent set of G, denoted by $\alpha(G)$ [18]. The algorithm that solves the Clique Cover problem can be modified slightly to solve the Independent Set problem in polynomial time [15]. We use these two properties of chordal graphs in the proof of the lemma that follows.

Lemma 3.4. *Let $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ be a* Left-Sided Terrain Guarding *instance where* LVis $\mathcal{G} \supseteq \mathcal{C}$. *One can decide, in polynomial time, if this is a true instance of the problem. If this instance is false, then one can find $U \subseteq \mathcal{C}$ in polynomial time such that $|U| = k + 1$ and $(T(V, E), n, k, \mathcal{G}, U)$ is a false instance.*

Proof. By Theorem 3.3, we know that the left visibility graph, say G_T, corresponding to $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ is chordal and that it is a true instance if, and only if, $(G_T(\mathcal{C}, E'), |\mathcal{C}|, k)$ is a true instance of the Clique Cover problem. Since the Clique Cover problem can be solved in polynomial time in chordal graphs, we can decide if $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ is a true instance of the Left-Sided Terrain Guarding problem in polynomial time.

Now, if $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ is false, then G_T cannot be covered by k many cliques. Thus, $\overline{\chi}(G_T) > k$. This implies that $\alpha(G_T) > k$. We compute a maximum sized independent set of G_T and let U be a subset of size $k+1$ of this independent set. Since G_T is chordal, this can be done in polynomial time. Clearly, U is an independent set of G_T. If there exists k-many guards in $\mathcal{G}$ which guards U, then there must exist one guard which sees at least two vertices of U. By construction of G_T, there must exist an edge between them. This contradicts the fact that U is an independent set of G_T and thus completes the proof of this lemma. □

We note that the above lemma holds for the right-sided version of the terrain guarding problem as well. The lemma stated and proved above generalizes Lemmas 4.8 and 4.9 of [3]. These were used to present a FPT algorithm with respect to the solution size for the Discrete Orthogonal Terrain Guarding problem in that paper. Algorithm 1 presents the procedure described in the

Algorithm 1: Guarding Terrains with Left Guards

Input: A Left-Sided Terrain Guarding instance $(T(V,E), n, k, \mathcal{G}, \mathcal{C})$ where LVis $\mathcal{G} \supseteq \mathcal{C}$
Result: Yes if $(T(V,E), n, k, \mathcal{G}, \mathcal{C})$ is a true instance; a $U \subseteq \mathcal{C}$ with $|U| = k+1$ such that $(T(V,E), n, k, \mathcal{G}, U)$ is a false instance otherwise

```
G_T(C, E') ⟵ ComputeVisGraph((T(V,E), n, k, G, C))
χ̄(G_T) ⟵ MinCliqueCover(G_T(C, E'))
if χ̄(G_T) ≤ k then
  | Output: Yes
end
else
  | I ⟵ MaxIndSet(G_T(C, E'))
  | U ⊆ I such that |U| = k + 1
  | Output: U
end
```

proof of Lemma 3.4. In this algorithm, we use well known subroutines to compute left visibility graphs of terrains (ComputeVisGraph), to find the smallest number of cliques required to cover a chordal graph (MinCliqueCover) and to find independent sets of maximum size of chordal graphs (MaxIndSet). The trivial algorithm to compute the visibility graph of a terrain runs in $\mathcal{O}(n^4)$ time. However, there are more complicated algorithms which run in quadratic time to compute visibility graphs [19]. The author refers the reader to [18] for greedy quadratic time algorithms which solve the Clique Cover and Independent Set problems.

We conclude this section by drawing parallels between Theorem 3.3 and the following result by Katz and Roisman [20].

Lemma 3.5. *Consider the* Annotated Orthogonal Terrain Guarding *instance* $(T(V,E), n, k, \mathscr{R}, \mathscr{C}_l)$ *and let* G_l *be the visibility graph corresponding to this instance. Then,* G_l *is chordal. Furthermore,* $(T(V,E), n, k, \mathscr{R}, \mathscr{C}_l)$ *is a true instance of the problem if, and only if,* $(G_l(\mathscr{C}_l, E'), |\mathscr{C}_l|, k)$ *is a true instance of the* Clique Cover *problem. The symmetric claim holds for the set of right convex vertices.*

Note that a vertex $v \in \mathscr{C}_l$ can only see a reflex vertex which is to its right (with the possible exception of the reflex vertex which just precedes it) [20]. Referring back to Figure 1b will make this observation straightforward. Furthermore, since $\mathscr{C} \cap \mathscr{R} = \emptyset$, the guards that are placed at these reflex vertices do not need to see themselves. Equivalently, a vertex $g \in \mathscr{R}$ which is to guard v needs to look only to its left. Thus, for guarding $\mathscr{C}_l$, we can consider $\mathscr{R}$ to be a set of left guards. Using a symmetric argument, we see that the guard set that is to guard the right convex vertices can be considered to be a set of right guards. Clearly, since Vis $\mathscr{R} \supseteq V$, $\mathscr{R}$ sees all of $\mathscr{C}_l$ and $\mathscr{C}_r$.

4 Natural Extensions

In this final section of the paper, we use the results obtained in Section 3 to solve two problems. We first discuss the LEFT-SIDED CONTINUOUS TERRAIN GUARDING problem in which we are required to guard the entire terrain (with the exception of the last vertex) using left guards which can be placed anywhere on the terrain. We first prove that it is enough to consider the case where the guards are placed on the vertices of the terrain T. We then construct a polynomially large ($\mathcal{O}(n^2)$-sized) set W of points with the following property: if a set of left guards see W, then it must see all of T. Hence, we can use Theorem 3.3 to solve the LEFT-SIDED CONTINUOUS TERRAIN GUARDING problem.

Finally, we extend and modify Theorem 3.3 and Lemma 3.4 to solve the LEFT-SIDED DOMINATING SET problem in terrain-like graphs (these terms are defined in Section 4.2). We prove that for a LEFT-SIDED TERRAIN GUARDING instance there exists an equivalent LEFT-SIDED DOMINATING SET instance in this subclass of graphs. Since the set of terrain-like graphs that correspond to terrains is a strict subset of the entire subclass of such graphs, the results obtained in this section generalize the results in Section 3.

As in Section 3, we deal exclusively with the left-sided versions of problems in this section. Right-sided versions of these problems and the results corresponding to them can be obtained using simple symmetry arguments.

4.1 Continuous Terrain Guarding

As mentioned briefly in Section 1, in the CONTINUOUS TERRAIN GUARDING problem, given a natural number k and a terrain T, we are required to determine if there is a set of points on the terrain of size at most k that guards the entire terrain. In this subsection, we define the LEFT-SIDED CONTINUOUS TERRAIN GUARDING problem. For an instance of this problem, we use discretization techniques similar to ones in [7] and [14] to construct an equivalent LEFT-SIDED TERRAIN GUARDING instance. Hence, by Theorem 3.3, we have a polynomial time algorithm to solve the LEFT-SIDED CONTINUOUS TERRAIN GUARDING problem.

Problem. LEFT-SIDED CONTINUOUS TERRAIN GUARDING: Let $T(V, E)$, where $V = \{v_1, v_2 \dots v_n\}$, be a terrain and $k \in \mathbb{N}$. Decide if there exists a $S \subset T$ with $|S| \leq k$ such that LVIS $S = T \setminus \{v_n\}$.

Since the last vertex of a terrain is the rightmost vertex of the terrain, no left guard can guard it. To exclude this trivial false instance, we only require S to guard $T \setminus \{v_n\}$ in the LEFT-SIDED CONTINUOUS TERRAIN GUARDING problem. We let $(T(V, E), n, k)$ denote an instance of this problem. For two points t_1 and t_2 on a terrain T where $t_1 \prec t_2$, we let T_{t_1,t_2} denote the part of the terrain that lies between these points. We now make the following simple observation. The proof of this observation is based on the one in [7].

Observation 4.1. Let $T(V, E)$ be a terrain and $S \subseteq T$. Then, there exists $S' \subseteq V$ with $|S'| \leq |S|$ such that LVIS $S' \supseteq$ LVIS S.

Proof. Let $g \in S$ be a guard placed in the interior of an edge of T and $t \in \text{LVIS } g$. Let (v_j, v_{j+1}) denote the edge within which g lies. Then, $t \prec g$ and does not lie below the line $\overleftrightarrow{v_j v_{j+1}}$. Since g sees t, no point in $T_{t,g}$ lies above $\overline{tg}$. Hence, no point in $T_{t,v_{j+1}}$ lies above $\overline{tv_{j+1}}$. This proves that $t \in \text{LVIS } v_{j+1}$. This is illustrated in Figure 5a. Hence, $\text{LVIS } v_{j+1} \supseteq \text{LVIS } g$. If, we construct S' from S by moving a guard in the interior of the edge to its right endpoint, then $|S'| \leq |S|$ and $\text{LVIS } S' \supseteq \text{LVIS } S$. □

By Observation 4.1, we can assume that the optimal left guard set which guards a terrain is a subset of the vertex set of that terrain. We now construct a finite subset W of a terrain $T(V,E)$, which we call the *witness set*, with the following property: for any $S \subseteq V$ with $\text{LVIS } S \supseteq W$, $\text{LVIS } S = T \setminus \{v_n\}$ where v_n is the last vertex of T.

Constructing W. Consider a vertex v of the terrain. Then, v sees at most n-many maximal "pieces" of the terrain to its left (refer to Figure 5b for an example). Let $W'(v)$ denote the set of endpoints of these pieces and let $P = \cup_{v \in V} W'(v)$. Let $\{p_1, p_2 \dots p_l\}$ be the points of P sorted according to how they appear on the terrain. For each j, where $1 \leq j < l$, let $T_j = T_{p_j, p_{j+1}}$. Pick some point $w_j \in T_j \setminus \{p_j, p_{j+1}\}$ for each j and let $W = \{w_1, w_2 \dots w_{l-1}\}$.

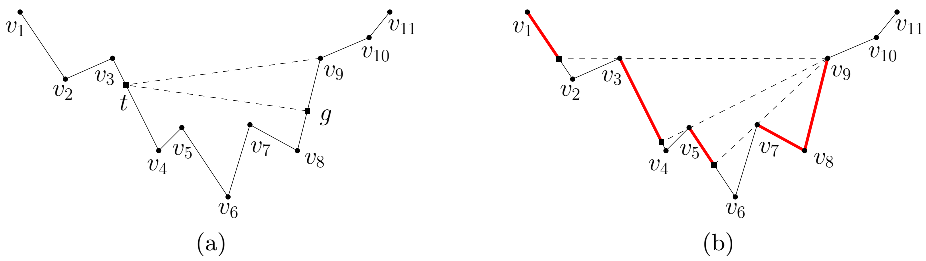

Fig. 5. Part (a) illustrates Observation 4.1. Here, g is a left guard on the interior of the (v_8, v_9) edge and t is a point that g sees. Then, v_9 sees t. In part (b) of this figure, the maximal pieces of the terrain seen by a left guard placed at v_9 are marked in red. The endpoints of these pieces which are not vertices are marked using black squares.

Observation 4.2. W is a witness set of $T(V,E)$ where $V = \{v_1, v_2 \dots v_n\}$.

Proof. Let $S \subseteq V$ such that $\text{LVIS } S \supseteq W$. We prove that $\text{LVIS } S = T \setminus \{v_n\}$. Let $w_j \in W$. By our assumption, there exists a left guard $g \in S$ which sees w_j. Since $S \subseteq V$, $g \in V$. Hence, by construction of P, $\text{LVIS } \supseteq T_j$. Since $\cup_{j:w_j \in W} T_j = T \setminus \{v_n\}$, $\text{LVIS } S = T \setminus \{v_n\}$. □

We construct a terrain $\bar{T}(\bar{V}, \bar{E})$ from $T(V,E)$ by setting $\bar{V} = V \cup W$. Hence, from Observations 4.1 and 4.2, we directly have the following lemma.

Lemma 4.3. *Let $(T(V,E), n, k)$ be an instance of the* Left-Sided Continuous Terrain Guarding *problem. Then, this instance is equivalent to the* Left-Sided Terrain Guarding *instance $(\bar{T}(\bar{V}, \bar{E}), |\bar{V}|, k, V, W)$ where W is the witness set of T and $\bar{V} = V \cup W$.*

Since $W \in \mathcal{O}(n^2)$, by Theorem 3.3 we have an $\mathcal{O}(n^4)$ algorithm to solve the Left-Sided Continuous Terrain Guarding problem.

4.2 Terrain-Like Graphs

Let $G(V,E)$ be an undirected graph. We say that u *dominates* v in G if $u \in N_G[v]$. Here, $N_G[v]$ denotes the closed neighbourhood of the vertex v in G. That is, $N_G[v] = \{u \mid (u,v) \in E\} \cup \{v\}$. For a subset S of V, we define $N_G[S]$ as $\cup_{v \in S} N_G[v]$. S is a *dominating set* of G if $N_G[S] = V$. In the classical Dominating Set problem, we are given an undirected graph and a natural number k and are asked if there exists a dominating set of size at most k. In this subsection, we study a variant of the Dominating Set problem in the family of *terrain-like* graphs. This family of graphs was first defined and used by Ashur *et al.* in [4].

Definition (Terrain-Like Graph). Let $H(V,F)$ be an undirected graph where $n = |V|$. H is said to be *terrain-like* if there exists an ordering $\{v_1, v_2 \dots v_n\}$ of V such that for all $\{v_i, v_j, v_k, v_l\} \subseteq V$ where $i < j < k < l$ and (v_i, v_k), (v_j, v_l) belong to the edge set, $(v_i, v_l) \in F$.

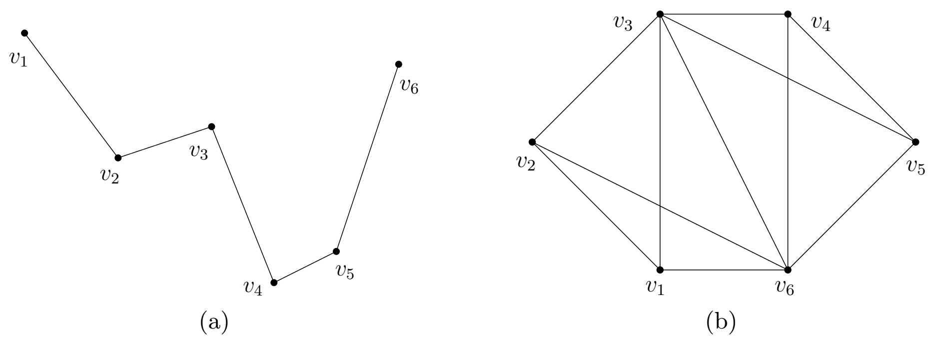

Fig. 6. Part (a) of this figure is a terrain $T(V,E)$ on 6 vertices. $\tilde{G}_T(V, \tilde{E})$, where $\tilde{E} = \{(u,v) \in V^2 \mid u \text{ sees } v\}$ is illustrated in (b). By Observation 4.4, $\tilde{G}$ is terrain-like.

An example of a terrain-like graph is given in Figure 6b. When working with a terrain-like graph, we fix an ordering of the vertices for which the above definition holds. For two vertices v_i and v_j of a terrain-like graph we say v_i *precedes* v_j, denoted by $v_i \prec v_j$ if $i < j$. By Lemma 2.1, the following observation is clear.

Observation 4.4. Let $T(V,E)$ be a terrain where $V = \{v_1, v_2 \ldots v_n\}$. Then, $\tilde{G}_T(V,\tilde{E})$, where $\tilde{E} = \{(u,v) \in V^2 \mid u \text{ sees } v\}$ and the vertices are ordered according to how they appear on terrain, is terrain-like. Moreover, if $\mathcal{G}, \mathcal{C} \subseteq V$, a subset S of $\mathcal{G}$ dominates $\mathcal{C}$ in $\tilde{G}$ if, and only if, S guards $\mathcal{C}$ in T.

Indeed, it is due to Observation 4.4 that terrain-like graphs are named as they are. However, not all terrain-like graphs arise out of terrains. For example, consider $H(V,F)$ where $V = \{v_1, v_2, v_3\}$ and $F = \{(v_1, v_2)\}$. For any terrain $T(V,E)$ defined on three vertices, the second vertex sees the other two vertices. Hence, $H(V,F)$ is not $\tilde{G}_T(V,\tilde{E})$ for any terrain T. We now define an analogue of the Left-Sided Terrain Guarding problem below for terrain-like graphs.

Problem. Left-Sided Dominating Set: Let $H(V,F)$ be a terrain-like graph, $k \in \mathbb{N}$ and $\mathcal{G}, \mathcal{C} \subseteq V$. Decide if there exists a $S \subseteq \mathcal{G}$ with $|S| \leq k$ such that for any $v \in \mathcal{C}$, there is a $g \in S$ such that g dominates v and $v \prec g$.

We let $(H(V,F), n, k, \mathcal{G}, \mathcal{C})$ denote an instance of the Left-Sided Dominating Set problem where $n = |V|$. As in Section 1, define the *left visibility graph* of such an instance as the undirected graph $G_H = (\mathcal{C}, F')$, where $F' = \{(u,v) \mid \text{there is a } g \in \mathcal{G}, \text{ where } u \prec g \text{ and } v \prec g, \text{ such that } u,v \in N_H[g]\}$. Since the proofs of Theorems 3.1 and 3.2 do not use any geometric properties of terrains, these proofs can be modified to prove the following results regarding the Left-Sided Dominating Set problem.

Theorem 4.5. *Let $(H(V,F), n, k, \mathcal{G}, \mathcal{C})$ be a* Left-Sided Dominating Set *instance where for all $v \in \mathcal{C}$, there is a $g \in \mathcal{G}$ such that $v \prec g$ and g dominates v. Then, this is a true instance of the problem if, and only if, $(G_H(\mathcal{C}, F'), |\mathcal{C}|, k)$ is a true instance of the* Clique Cover *problem where G_H, the left visibility graph of H, is a chordal graph.*

Lemma 4.6. *Let $(H(V,F), n, k, \mathcal{G}, \mathcal{C})$ be a* Left-Sided Dominating Set *instance where for all $v \in \mathcal{C}$, there is a $g \in \mathcal{G}$ such that $v \prec g$ and g dominates v. One can decide, in polynomial time, if this is a true instance of the problem. If this instance is false, then one can find $U \subseteq \mathcal{C}$ in polynomial time such that $|U| = k+1$ and $(H(V,F), n, k, \mathcal{G}, U)$ is a false instance.*

The above two results directly imply that Algorithm 1 can be modified to solve Left-Sided Dominating Set. By Observation 4.4, a Left-Sided Terrain Guarding instance $(T(V,E), n, k, \mathcal{G}, \mathcal{C})$ is equivalent to the Left-Sided Dominating Set instance $(\tilde{G}_T(V,\tilde{E}), n, k, \mathcal{G}, \mathcal{C})$. Hence, Theorem 4.5 and Lemma 4.6 are stronger results than Theorem 3.3 and Lemma 3.4 respectively.

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